Selected Topics on the weak topology of Banach spaces

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E - Banach space, $E_w := (E, w(E, E'))$, B_w – the closed unit ball with the weak topology, K – compact space.

1. **Cosmic spaces, $\aleph_0$-spaces, $\aleph$-spaces and σ-spaces, topological characterizations.**
2. **Networks for spaces E_w; general case.**
3. **Networks for spaces E_w where $E := C(K)$.**
4. **Generalized metric concepts for spaces E_w and B_w; a bit of history.**
5. **k_R-spaces, Ascoli and stratifiable spaces E_w and B_w.**
6. **Ascoli spaces $C_p(X)$ and $C_k(X)$.**
7. **Open problems.**

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An excellent monograph of renorming theory up to 1993 is: [Deville, Godefroy, Zizler] **Smoothness and renormings in Banach spaces**, Pitman Monographs and Surveys in Pure and Applied Mathematics.

For a Banach space E the following are equivalent: (i) Every E_w-compact set is E_w-metrizable. (ii) Every E_w-compact set is contained in a separable subset of E_w. (iii) E_w is the image under a compact-covering map of a metric space F.

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Theorem 1 (Michael)

For a regular space X the following are equivalent.

(i) *X is the image under a compact-covering map of a separable metric space.*

(ii) *There exists a countable family $\mathcal{D}$ (countable k-network) of subsets in X such that for each open set U in X and compact $K \subset U$ there exists $D \in \mathcal{D}$ with $K \subset D \subset U$.*

Few definitions and facts. X – regular.

1. X is an $\aleph_0$-space if X has a countable k-network [Michael]. Any metric separable X is an $\aleph_0$-space.
2. X is cosmic if X has a countable network.
3. X is cosmic iff X is a continuous image of a metric separable space.
4. X is an $\aleph$-space if X has a σ-locally finite k-network [0'Meara]. Any metric space is an $\aleph$-space, compact sets in $\aleph$-spaces are metrizable, see Gruenhage's works.
5. X is an $\aleph_0$-space iff X is a Lindelöf $\aleph$-space.
6. X is a σ-space if X has a σ-locally finite network [Okuyama] (eq., σ-discrete network [Siwiec-Nagata]).

Theorem 2 (0'Meara-Foged)

If X is an $\aleph_0$-space and Y is a (paracompact) $\aleph$-space, then $C_k(X, Y)$ with the compact-open topology is an (paracompact) $\aleph$-space. Hence, if X is separable metric and Y is metric, then $C_k(X, Y)$ is paracompact.

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Theorem 4 (Gabriyelyan-K.-Kubiś-Marciszewski)

An $\aleph$-space X is metrizable iff X is Fréchet-Urysohn with α_4-property. Hence a Fréchet-Urysohn topological group is metrizable iff it is an $\aleph$-space.

$$
\begin{array}{ccccc}
\text{separable metrizable} & \Longrightarrow & \aleph_0\text{-space} & \Longrightarrow & \text{cosmic} \\
\Downarrow & & \Downarrow & & \Downarrow \\
\text{metrizable} & \Longrightarrow & \text{strict } \aleph\text{-space} & \Longrightarrow & \text{strict } \sigma\text{-space} \\
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Let $\mathbf{M} \subseteq \mathbb{N}^{\mathbb{N}}$ and $\mathcal{U} = \{U_\alpha : \alpha \in \mathbf{M}\}$ be an **M**-decreasing family of subsets of a set X. Define the countable family $\mathcal{D}_{\mathcal{U}}$ of subsets of X by

$$\mathcal{D}_{\mathcal{U}} := \{D_k(\alpha) : \ \alpha \in \mathbf{M}, \ k \in \mathbb{N}\}, \text{ where } D_k(\alpha) := \bigcap_{\beta\in I_k(\alpha)\cap\mathbf{M}} U_\beta,$$

$\mathcal{U}$ satisfies *condition* (**D**) if $U_\alpha = \bigcup_{k\in\mathbb{N}} D_k(\alpha)$, $\alpha \in \mathbf{M}$.

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$\mathcal{U}$ satisfies *condition* (**D**) if $U_\alpha = \bigcup_{k\in\mathbb{N}} D_k(\alpha)$, $\alpha \in \mathbf{M}$.

(X, τ) has a *small base* if there exists an **M**-decreasing base of τ for some $\mathbf{M} \subseteq \mathbb{N}^{\mathbb{N}}$ [Gabriyelyan-K.].

Theorem 5 (Gabriyelyan-K.)

(i) *X is cosmic iff X has a small base* $\mathcal{U} = \{U_\alpha : \alpha \in \mathbf{M}\}$ *with condition* $(\mathbf{D})$*. In that case the family* $\mathcal{D}_\mathcal{U}$ *is a countable network in* X*.*

(ii) *X is an $\aleph_0$-space iff X has a small base* $\mathcal{U} = \{U_\alpha : \alpha \in \mathbf{M}\}$ *with condition* $(\mathbf{D})$ *such that the family* $\mathcal{D}_\mathcal{U}$ *is a countable k-network in* X*.*

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Corollary 6

Let G be a Baire topological group. Then G is cosmic iff G is metrizable and separable.

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Theorem 7 (Schlüchtermann-Wheeler)

The following are equivalent for a Banach space E.

(i) B_w *is Fréchet–Urysohn.*

(ii) B_w *is sequential.*

(iii) B_w *is a k-space, i.e.* $P \subset B_w$ *is closed in* B_w *if* $P \cap K$ *is closed in* K *for all compact* $K \subset B_w$.

(iv) E *contains no isomorphic copy of* ℓ_1.

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(iv) *E contains no isomorphic copy of ℓ_1.*

Theorem 8 (Schlüchtermann-Wheeler)

If E is a Banach space, then E_w is a k-space iff $\dim(E) < \infty$.

Theorem 9 (Schlüchtermann-Wheeler)

The following conditions are equivalent for a Banach space E.

(i) B_w *is (separable) metrizable.*

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For a Banach space $E := C(K)$ the space E_w is an $\aleph$-space iff E_w is an $\aleph_0$-space iff K is countable.

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Hence, the assumption on $C(K)_w$ to have a σ-locally finite k-network is much to strong.

Theorem 14 (Gabriyelyan-K.-Kubiś-Marciszewski)

Let E be a Banach space not containing a copy of ℓ_1. The following conditions are equivalent:

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Corollary 15

If E is separable and does not contain ℓ_1, then E_w is an $\aleph_0$-space iff E' has a w^*-Kadec norm.

Theorem 16 (Gabriyelyan-K.-Kubiś-Marciszewski)

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Hence $(\ell_1(\mathbb{R}))_w$ is an $\aleph$-space which is not an $\aleph_0$-space and $(\ell_1(\mathbb{R}))_w$ is not normal. Last claim follows from:

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Theorem 17 (Reznichenko)

Let E be a Banach space. Then E_w is Lindelöf iff E_w is normal iff E_w is paracompact.

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Let Γ be an infinite set and $E := \ell_p(\Gamma)$ with $1 < p < \infty$. Then $\psi(E_w) \geq |\Gamma|$, where $E_w := (E, \sigma(E, E'))$. Hence $\ell_p(\Gamma)_w$ are not σ-spaces for any uncountable Γ. More: E_w for any nonseparable weakly Lindelöf E is not a σ-space.

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How to describe σ-spaces $C(K)_w$? Let's recall the concept of *descriptive Banach spaces*.

Theorem 19 (M-O-T-V-Hansell)

E is descriptive [Hansell] (i.e. E has a norm-network which is σ-isolated in E_w) iff E has the JNR-property iff E_w has a σ-isolated network.

E has *JNR* iff for any $\epsilon > 0$ there is a sequence (E_n^ϵ) covering E such that for any $n \in \mathbb{N}$ and any $x \in E_n^\epsilon$ there is an w-open neighbourhood $x \in U$ with $diam(U \cap E_n^\epsilon) < \epsilon$.

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Concrete spaces $C(K)$ with Kadec renorming: K - dyadic compacta, compact linearly ordered spaces, Valdivia compacta (hence Corson compacta), all "cubes" $[0,1]^\kappa$, AU-compacta....

$C(K)$ has JNR_C-property ($= C(K)$ has JNR-property $+$ $C_p(K)$ is perfect) iff there exists a σ-discrete family in $C_p(K)$ which is a network in $C(K)$ [Marciszewski-Pol].

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Concrete K: separable dyadic compacta, separable compact linearly ordered spaces.... [M.-P.]. **Then** $C_p(K)$ **and** $C(K)_w$ **are** σ**-spaces** (not $\aleph$-spaces).

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E is descriptive $\not\Rightarrow$ E_w is a σ-space.

Take $E := C(K)$ with $K := [0, \omega_1]$. E is descriptive, so E_w has a σ-isolated network, E_w does not admit a σ-discrete network (since E_w has uncountable pseudocharacter). Another example K separable: $C(K(\omega^{<\omega}))$ over AU-compact $K(\omega^{<\omega}) := \omega^{<\omega} \cup \omega^{\omega} \cup \{\infty\}$ [M.-P. 2009]:

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$C(\beta\mathbb{N})$ not descriptive. $C_p(\beta\mathbb{N})$, $C_p(\beta\mathbb{N} \setminus \mathbb{N})$ are not σ-spaces (separ. compact K are continuous images of $\beta\mathbb{N}$ ($\beta\mathbb{N} \setminus \mathbb{N}$)).

It is consistent with ZFC: there is a compact separable scattered space K such that $C(K)$ has no Kadec renorming and $C_p(K)$ is not a σ-space. [M.-P.]

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Problem 20 (M-O-T-V)

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Problem 22

Describe (separable) compact K for which $C(K)_w$ is a σ-space.

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If $C_p(X)$ is angelic then $C_p(X)$ is a $k_{\mathbb{R}}$-space iff it is a $s_{\mathbb{R}}$-space.

X is an **Ascoli space** if each compact $K \subset C_k(X)$ is evenly continuous [Banakh-Gabriyelyan], i.e. for $\psi : X \times C_k(X) \to \mathbb{R}$, $\psi(x, f) := f(x)$, the map $\psi | X \times K$ is jointly continuous.

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Theorem 24 (Gabriyelyan-K.-Plebanek)

E_w is Ascoli iff E is finite-dimensional.

Problem 25

Does there exist a Banach space E containing a copy of ℓ_1 such that B_w is Ascoli or even a $k_{\mathbb{R}}$-space?

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Theorem 26 (Gabriyelyan-K.-Plebanek)

The following are equivalent for a Banach space E.

(i) B_w is Ascoli, i.e. B_w embeds into $C_k(C_k(B_w))$;

(ii) B_w is a $k_{\mathbb{R}}$-space;

(iii) B_w is a $s_{\mathbb{R}}$-space;

(iv) E does not contain a copy of ℓ_1.

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Theorem 27 (Gabriyelyan-Grebik-K.-Zdomskyy)

Let X be a Čech-complete space. Then:

(i) *If $C_p(X)$ is Ascoli, then X is scattered.*

(ii) *If X is scattered and stratifiable, then $C_p(X)$ is an Ascoli space.*

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Corollary 28

Let X be a completely metrizable space. Then $C_p(X)$ is Ascoli iff X is scattered.

Corollary 29

(A) For Čech-complete Lindelöf X, the following are equiv.

(i) $C_p(X)$ *is Ascoli.*

(ii) $C_p(X)$ *is Fréchet–Urysohn.*

(iii) $C_p(X)$ *is a* $k_{\mathbb{R}}$*-space.*

(iv) X *is scattered.*

(B) If X is locally compact, then $C_p(X)$ is Ascoli iff X scattered.

Theorem 30 (Gabriyelyan-Grebik-K.-Zdomskyy)

For paracompact of point-countable type X the following are equiv.

(i) *X is locally compact.*

(ii) *$C_k(X)$ is a $k_{\mathbb{R}}$-space.*

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(i) The first claim follows from the local compactness and the scattered property of $[0,\omega_1)$.

(ii) Assume $E := C_p([0,\omega_1))$ is a $k_{\mathbb{R}}$-space. Since $[0,\omega_1)$ is pseudocompat, E is dominated by a Banach topology. Hence E is angelic, so every compact set in E is Fréchet-Urysohn. Therefore E is a $s_{\mathbb{R}}$-space, and then $[0,\omega_1)$ is realcompact, a contradiction.

When E_w is stratifiable?

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X is stratifiable iff to each open $U \subset X$ one can assign a continuous function $f_U : X \to [0,1]$ such that $f^{-1}(0) = X \setminus U$, and $f_U \leq f_V$ whenever $U \subset V$ [Borges].

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If X is stratifiable, then X is separable iff X is Lindelöf iff X has countable network.

X stratifiable, $A \subset X$ closed, then there is a continuous linear extender $e : C_k(A) \to C_k(X)$, $e(f)|A = f$ for any $f \in C(A)$, (Dugundji extenstion property) [Borges].

X is Polish $\Rightarrow C_k(X)$ is stratifiable [Gartside-Reznichenko]. They conjectured: If X is separable metrizable and $C_k(X)$ is stratifiable, then X is Polish.

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Many examples of nonmetrizable stratifiable LCS are provided by [Shkarin].

Theorem 31 (Gartside)

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Theorem 32 (Corson-Lindenstrauss)

For B_w of a nonseparable Hilbert space E and any $0 < \alpha < \beta$ there exists no weak-continuous retraction $r : \beta B_w \to \alpha B_w$, i.e. a map r such that $r(x) = x$ for every $x \in \alpha B_w$.

Theorem 33 (Aviles-Marciszewski)

For a nonseparable Hilbert space E and any $0 < \alpha < \beta$ there is no continuous extender $T : C(\alpha B_w) \to C(\beta B_w)$.

Easier approach for a weaker result: If E is weakly Lindelöf nonseparable, then B_w is not a σ-space (since E_w is not a σ-space). Hence B_w is not stratifiable.

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Problem 34

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Problem 34

Characterize those Banach spaces E for which B_w is stratifiable (has the Dugundji extension property).

Problem 35

Is the ball B_w a stratifiable space for $E := JT$?