On topological properties of Fréchet spaces with the weak topology

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Outline

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Cosmic and $\aleph_0$- spaces

Definition

A family $\mathcal{N}$ of subsets of a topological space X is called a network in X if, whenever $x \in U$ with U open in X, then $x \in N \subset U$ for some $N \in \mathcal{N}$.
X is called cosmic, if X is a regular space with a countable network.

Definition

A family $\mathcal{N}$ of subsets of X is called a k-network in X if, whenever $K \subset U$ with K compact and U open in X, then $K \subset \bigcup \mathcal{F} \subset U$ for some finite family $\mathcal{F} \subset \mathcal{N}$.
X is an $\aleph_0$-space, if X is a regular space with a countable k-network.

Cosmic and $\aleph_0$- spaces

The class of $\aleph_0$-spaces (Michael) is the most near extension of the class of separable metrizable spaces.

Theorem

A regular space is a cosmic (resp. $\aleph_0-$) space if and only if X is a continuous (resp. continuous compact-covering) image of a separable metric space.

Example (E separable Banach space (Michael))

1. The dual E'_β is weakly* $\aleph_0$-space.
2. If E'_β is also separable then E is weakly $\aleph_0$-space.

Recall E'_β separable implies $\ell_1 \not\subset E$.

Two more examples

Example (Related with Schur property)

If (E, ξ) be a separable Banach space with the Schur property then $(E, \sigma(E, E'))$ is an $\aleph_0$-space.

Proof.

Every weak null-sequence in E converges in ξ. By Šmulian (Dieud-Schwartz) $\sigma(E, E')$ and ξ have the same compact sets. □

Whence $E := \ell_1 \times \ell_2$ is a weakly $\aleph_0$-space, but E does not have the Schur property and its strong dual is nonseparable.

Example (Corson)

If X is a compact metric space, then $C(X, \mathbb{R})$ is an $\aleph_0$-space in the weak topology if and only if X is countable.

A Michael question

Question

When a separable Banach space E is $\aleph_0$-space in $\sigma(E, E')$?

Clearly the answer is "yes" if $E = F \times G$, F has Schur property and G'_β is a separable Banach space.

Solution (when $\ell_1 \not\subset E$)

E is a weakly $\aleph_0$-space if and only if E'_β is separable.
In general, let E be a Fréchet space satisfying the Heinrich density condition and not containing ℓ_1. E is a weakly $\aleph_0$-space if and only if E'_β is separable.

JT is not a weakly $\aleph_0$-space, $\Longrightarrow$ class of weakly $\aleph_0$ lcs is not closed under quotients (sep B space is quotient of ℓ_1).

Steps for solution when $\ell_1 \not\subset E$

Proposition

If a locally convex space (lcs) E is weakly $\aleph_0$-space then E'_β is trans-separable if and only if every bounded set in E is Fréchet-Urysohn in the weak topology of E.

Proposition (extension Barroso, Kalenda, Lin)

Let E be a Fréchet lcs, then:

1. *If E'_β is separable it follows that E is a weakly $\aleph_0$-space.*
2. *If E is a weakly $\aleph_0$-space not containing ℓ_1, then E'_β is trans-separable.*

Outline

2 Weakly $\aleph_0$-spaces not containing a copy of ℓ_1

- Weakly $\aleph_0$-spaces and trans-separability
- Fréchet version of a Michael weakly $\aleph_0$ theorems
- Examples of cosmic and $\aleph_0$-spaces

Fréchet-Urysohn and Strongly Fréchet-Urysohn spaces

Definition

A topological space X has the *property* (α_4) if for any $\{x_{m,n} : (m,n) \in \mathbb{N} \times \mathbb{N}\} \subset X$ with $\lim_n x_{m,n} = x \in X$, $m \in \mathbb{N}$, there exists a sequence $(m_k)_k$ of distinct natural numbers and a sequence $(n_k)_k$ of natural numbers such that $\lim_k x_{m_k,n_k} = x$.

Definition

A topological space X is strongly Fréchet-Urysohn if for every $x \in X$ and for each decreasing family (A_n) of X with $x \in \bigcap_n \overline{A_n}$, there are $x_n \in A_n$ $(n \in \mathbb{N})$ with $\lim_n x_n = x$.

F-U and $(\alpha_4) \Longrightarrow$ strongly F-U.

Trans-separable spaces

Definition

$(X, \mathcal{N})$ is trans-separable if $\forall N \in \mathcal{N}$ there exists contable $Q \subset X : X = \bigcup_{x \in Q} U_N(x)$.

- metrizable trans-separable $\Longrightarrow$is separable.
- A lcs E is trans-separable if and only if for each neighbourhood of zero U in E there exists a countable subset N of E with $E = N + U$.
- Note that a lcs E does not contain ℓ_1 provided the strong dual E' is trans-separable.

Lemma

The strong dual of a lcs E is trans-separable if and only if every bounded set in E is metrizable in the weak topology $\sigma(E, E')$.

Strongly Fréchet-Urysohn

Lemma

E tvs s.t. $\forall$bounded subset is Fréchet-Urysohn. Then $\forall$ bounded B has property $(\alpha_4) \implies E$ is strongly F-U.

Proof.

(tvs) $0 \neq x_{m,n} \in B$ with $\lim_n x_{m,n} = 0$, $\forall m \in \mathbb{N}$. Let $(v_m)_m \subset B\backslash\{0\}$, $\lim_m v_m = 0$ and

$y_{m,n} := x_{m,n+m}$ (if $x_{m,n+m} = v_m$) or $:= x_{m,n+m} - v_m$ if $x_{m,n+m} \neq v_m$

$0 \in \overline{\{y_{m,n} : (m,n) \in \mathbb{N}^2\}} \implies \exists \lim_k y_{p_k,s_k} = 0$.
$(p_k)_k$ bounded, $(s_k)_k$ is unbounded ($p_k = p$ and $s_k < s_{k+1}$) then

$$v_p = (y_{p_k,s_k} \quad \text{or} \quad x_{p,s_k+p} - y_{p_k,s_k}) = 0 \ \ \text{(contradiction)}.$$

Strongly Fréchet-Urysohn

Lemma

E tvs s.t. $\forall$bounded subset is Fréchet-Urysohn. Then $\forall$ bounded B has property $(\alpha_4) \Longrightarrow$ E is strongly F-U.

Proof: Last part.

$(p_k)_k$ bounded, $(s_k)_k$ bounded ($p_k = p$ and $s_k = s$) then
$y_{p,s} := \lim_k y_{p_k,s_k} = 0$ (contradiction).
So $(p_k)_k$ is unbounded.

□

Weakly $\aleph_0$-lcs and trans-separability

Proposition

Let E be a lcs which is weakly $\aleph_0$-space. E'_β is trans-separable if and only if every bounded set in E is Fréchet-Urysohn in the weak topology of E.

Proof.

We know that E'_β trans-separable $\Longrightarrow$ every bounded set in E is metrizable in $\sigma(E, E')$.
If every bounded subset B of E is F-U in $\sigma(E, E')$ then every bounded subset B of E is strongly Fréchet-Urysohn in $\sigma(E, E')$.
As a subspace of the $\aleph_0$-space $(E, \sigma(E, E'))$, the set B is also an $\aleph_0$-space, whence (Michael) B is second countable, hence metrizable. Therefore E' is trans-separable. $\square$

Ruess theorem on Rosenthal property

Definition

A lcs E will be said to have the Rosenthal property if every bounded sequence in E
(R_1) either has a subsequence which is Cauchy in the weak topology $\sigma(E, E')$,
(R_2) or has a subsequence which is equivalent to the unit vector basis of ℓ_1.

Proposition (Ruess, recently)

Every sequentially complete lcs E whose bounded subsets are metrizable has the Rosenthal property.

Example: $(E', \beta(E', E))$, with E metrizable lcs with *Heinrich density condition*

Cascales-Orihuela spaces of class $\mathfrak{G}$

A covering $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ of E is a *resolution* if $A_\alpha \subset A_\beta$ whenever $\alpha \leq \beta$.

Definition

A lcs $E \in \mathfrak{G}$ if E' admits a resolution $\{A_\alpha : \alpha \in \mathbb{N}\}$ (called a $\mathfrak{G}$-representation for E) s. t. $\forall$ sequence in $\forall A_\alpha$ is equicontinuous

(LM)-spaces, (DF)-spaces $\in \mathfrak{G}$ (which is stable by subspaces, quotients, countable direct sums and products).

Definition

A base $\mathcal{U} = \{U_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ of E-neighborhoods of zero is a $\mathfrak{G}$-*base* if $\mathcal{U}$ is $\mathbb{N}^{\mathbb{N}}$-decreasing.

A quasibarrelled E has a $\mathfrak{G}$-base $\Longleftrightarrow$ $E \in \mathfrak{G}$.

Extension of a result of Barrroso, Kalenda and Lin.

Lemma

$E \in \mathfrak{G}$ with Rosenthal property (R_1). Each bounded, separable $B \subset E$ is Fréchet-Urysohn in the weak topology.

Proof.

As span$\{B\}$ is separable we may assume that E is separable, whence E'_σ admits a coarser metrizable lct and (Šmulyan) $\exists$ $\sigma(E', E)$-compact resolution $\{\overline{V_\alpha}^{\sigma(E',E)} : \alpha \in \mathbb{N}^{\mathbb{N}}\}$.
By Talagrand $E'_{\sigma*}$ is analytic, i.e. $\exists$ a continuous surjection $G : \mathbb{N}^{\mathbb{N}} \to E'_{\sigma*}$ which generates the immersion

$$H : E_\sigma \to C(\mathbb{N}^{\mathbb{N}}), \quad H(x)(\alpha) = G(\alpha)(x), \ \ x \in E, \alpha \in \mathbb{N}^{\mathbb{N}}.$$

$H(B)$ has $(R_1) \Longrightarrow$ (Kalenda) $H(B)$ F-$U \Longrightarrow B$ F-U. □

An equivalence for $\mathfrak{G}$-quasibarrelled spces

Lemma

$E \in \mathfrak{G}$ quasibarrelled with Rosenthal prop. For $\sigma(E, E')$:
(i) any E-bounded is Fréchet-Urysohn iff
(ii) any E-bounded sequence has a Cauchy subsequence.

Proof.

$(i) \Rightarrow (ii)$: (Barroso, Kalenda, Lin). Let $(x_n)_n$ ℓ_1-sequence E and $T_0 : \ell_1^0 \to X$ the natural inmersion.
$0 \in \overline{S}^{weak}$($S$ unit ℓ_1^0-sphere) and no sequence in S converges to 0 (Schur). $0 \in \overline{T_0(S)}^{weak}$ no sequence in $T_0(S)$ converges to 0.
$(ii) \Rightarrow (i)$: $x \in \overline{B}^{\sigma(E,E')}$, bounded $B \subset (E, \sigma(E, E'))$ (countable tightness CKS)$\Longrightarrow$ $\exists$countable $C \subset B$ such that $x \in \overline{C}^{\sigma(E,E')}$.
Previous lemma applies. □

A particular case of the equivalence

Corollary

Let E be the strong dual of a metrizable lcs with the Heinrich density condition. T.f.a.e:

1. *Any E-bounded is $\sigma(E, E')$-Fréchet-Urysohn.*
2. *Any E-bounded sequence has a $\sigma(E, E')$-Cauchy subsequence.*
3. *E does not contain ℓ_1.*

Proof.

Every bounded set in E is metrizable and E is a barrelled, complete (DF)-space, so $E \in \mathfrak{G}$.
E has Rosenthal property (Ruess).
Apply previous Lemma and Rosenthal property definition. □

Fréchet version of a Michael weakly $\aleph_0$ theorems

Theorem

Let E be a Fréchet lcs and E'_β be its strong dual. Then

1. *If $\ell_1 \not\subset E$, weakly $\aleph_0$-space $\Longrightarrow$ E'_β trans-separable.*
2. *E'_β separable $\Longrightarrow$ E weakly $\aleph_0$-space.*

Proof.

Note that by Ruess E has Rosenthal property. Hence if 1 we have (by preceding lemma) that every bounded set in E is Fréchet-Urysohn in $\sigma(E, E')$.
But in a lcs weakly $\aleph_0$-space E every bounded set in E is Fréchet-Urysohn if and only if E'_β is trans-separable. □

It holds for $E =$ Strong dual of a metrizable lcs with H.d.c.

Fréchet version of a Michael weakly $\aleph_0$ theorems

Theorem

Let E be a Fréchet lcs and E'_β be its strong dual. Then

1. *If $\ell_1 \notin E$, weakly $\aleph_0$-space $\Longrightarrow$ E'_β trans-separable.*
2. *E'_β separable $\Longrightarrow$ E weakly $\aleph_0$-space.*

Proof.

If E'_β is separable then E'_β is barrelled complete (DF)-space. Let $(U_n)_n \downarrow$ base 0-E-neigh. $\{U_\alpha = \bigcap_k \alpha_k U_k : \alpha \in \mathbb{P}\}$ is a bounded res. swallowing E-bounded subset, $\{U_\alpha^\circ : \alpha \in \mathbb{P}\}$ is a $\mathfrak{G}$-base in E'_β and $\{U_\alpha^{\circ\circ} : \alpha \in \mathbb{P}\}$ is a $E''_\sigma := (E'', \sigma(E'', E')$ compact resolution swallowing compact subsets (E'_β barrelled!). E'_β is separable$\Longrightarrow$ E''_{σ^*} is submetrizable $\Longrightarrow$ (COT Th. 3.6) E''_{σ^*} is $\aleph_0$-space $\Longrightarrow$ E weakly $\aleph_0$-space. □

A symmetric property

Proposition

Let E be a lcs. $(E, \sigma(E, E'))$ is cosmic if and only if $(E', \sigma(E', E))$ is cosmic.

Proof.

$(E, \sigma(E, E')) \subset C_p(E', \sigma(E', E))$,
$(E', \sigma(E', E)) \subset C_p(E, \sigma(E, E'))$,
and (Michael)
$(E, \sigma(E, E'))$ (cosmic)$\Longrightarrow C_p(E, \sigma(E, E'))$ (cosmic)
$(E', \sigma(E', E))$ (cosmic)$\Longrightarrow C_p(E', \sigma(E', E))$ (cosmic) □

Examples of weakly $\aleph_0$ spaces.

Let E be a lcs with strong dual $(E', \beta(E', E))$ separable.

Proposition

E is a weakly $\aleph_0$-space if E verifies one of the following conditions

1. *If E is metrizable*
2. *If E is a (DF)-space*
3. *If E is a strict (LF)-space*

Examples of weak* $\aleph_0$-spaces

Let E be a separable lcs.

Proposition

$(E', \sigma(E', E))$ is an $\aleph_0$-space when E verifies

1. *E is metrizable and barrelled.*
2. *E is a (LF)-space.*

Corollary

The space of distributions $D'(\Omega)$ is a weakly $\aleph_0$-space.

Examples of $\aleph_0$ $C_c(X)$ spaces.

Let X be a Čech-complete space.

Proposition

X is Polish if and only one of the following conditions hold:

1. $C_c(X)$ is an $\aleph_0$-space.
2. $C_c(X)$ is a cosmic space.
3. $C_c(X)$ is hereditarily separable.

Examples of $C_c(X)$ with $\aleph_0$ weak*-dual.

Let X be a Čech-complete Lindelöf space.

Proposition

X is Polish if and only if one of the following conditions hold:

1. $C_c(X)$ is a weakly cosmic space
2. $C_c(X)$ is separable
3. The weak*-dual of $C_c(X)$ is $\aleph_0$-space
4. The weak*-dual space of $C_c(X)$ is cosmic

Outline

3 Michael question and Heinrich density condition

A property concerning uniformities.

Let τ_1 and τ_2 be two lc topologies in E. If M is an abs convex subset of E and $\tau_1|_M = \tau_2|_M$ then the uniformities $\mathcal{N}_i$, $i = 1, 2$, genrerated by the l.c. structure coincide on M. We need the following refinement of this well known property.

Proposition

Let (E, τ) be a lcs and $\mathcal{N}$ be the uniformity on E generated by the locally convex structure of E. Let $A \subset E$ be an absolutely convex bounded subset of E such that the set $(4A, \tau|_{4A})$ is metrizable. Then there exists a metric d on $4A$ such that

1. $d(x - y, 0) = d(x, y)$ *for all* $x, y \in 4A$ *with* $x - y \in 4A$,
2. *the topology generated by* d *on* $2A$ *coincides with* $\tau|_{2A}$,
3. *the* d*-uniformity* $\mathcal{M}$ *on* A *and* $\mathcal{N}$ *coincide on* A.

Trans-separability and separability

Corollary

Let (E, τ) be a lcs having a sequence $\{Q_n\}_{n\in\mathbb{N}}$ of absolutely convex bounded sets covering E such that $(Q_n, \tau|_{Q_n})$ is metrizable for every $n \in \mathbb{N}$. Then E is trans-separable if and only if E is separable.

Proof.

Applying preceding proposition to $A_n = 4^{-1}Q_n$, $n \in \mathbb{N}$, we obtain that the trans-separable uniformity on A_n is metrizable, so A_n is separable. Thus $E = \bigcup_n A_n$ is separable. The converse is trivial. □

Michael question for E with H.d.c. and $l_1 \not\subset E$

Let E be a Fréchet lcs satisfying the Heinrich density condition and $\ell_1 \not\subset E$ (in particular, E Banach space with $\ell_1 \not\subset E$).

Theorem

E is a weakly $\aleph_0$-space if and only if E'_β is separable.

Proof.

E'_β is a (DF)-space with a fundamental sequence $(Q_n)_n$ of absolutely convex bounded metrizable subsets (by H.d.c.).
E (Fréchet, $\ell_1 \not\subset E$) weakly $\aleph_0$-space $\Longrightarrow$ E'_β trans-separable.
Corollary $\Longrightarrow$ E'_β is separable.
E'_β is separable (E Fréchet) $\Longrightarrow$ E is a weakly $\aleph_0$-space.

□

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