Selected topics on the weak topology of Banach spaces

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1. **The weak topology of (WCG) Banach spaces.**
2. **More about the weak topology of $C(K)$ spaces.**
3. **Generalized metric concepts.**
4. **Ascoli spaces.**
5. **Networks for the weak topology of Banach spaces.**
6. **Networks and renormings.**
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The weak topology of (WCG) Banach spaces.

1. Let E be a Banach space and E_w the space E endowed with the weak topology $w := \sigma(E, E')$. E'_{w^*} denotes the weak*-dual with the topology $w^* := \sigma(E', E)$. Let B_w be the closed unit ball in E with the weak topology.
2. Let X be a completely regular Hausdorff space. Let $C_p(X)$ and $C_c(X)$ be the spaces of all real-valued continuous functions on X endowed with the pointwise topology and the compact-open topology, respectively.
3. Clearly
$$E_w \hookrightarrow C_p(E'_{w^*}), \quad E'_{w^*} \hookrightarrow C_p(E_w).$$
4. **Corson** 1961 started a systematic study of certain topological properties of the weak topology of Banach spaces.

1. This line of research provided more general classes such as reflexive Banach spaces, **Weakly Compactly Generated Banach spaces** ((WCG) shortly) and the class of weakly K-analytic and weakly K-countably determined Banach spaces. (**Corson, Amir, Lindenstrauss, Talagrand, Preiss,**....). Still active area of research, for example the modern renorming theory deals also with spaces E_w.
2. Another line of such a research was essentially related with the concept of networks and k-networks for the spaces E_w.
3. E is (WCG) if E admits a weakly compact set with dense linear span in E.
4. $c_0(\Gamma)$ is (WCG) not being separable nor reflexive for uncountable Γ.
5. If E is (WCG), there exists a continuous linear injective map from E into $c_0(\Gamma)$ (**Amir-Lindenstrauss**).

Theorem 1 (Reznichenko)

The following conditions for a Banach space E are equivalent:

1. E_w *is Lindelöf.*
2. E_w *is normal.*
3. E_w *is paracompact.*

Theorem 2 (Talagrand-Preiss)

E_w *is Lidelöf (even K-analytic) for every (WCG) Bnach space* E*. Moreover,* $\mathit{dens}\, E_w = \mathit{dens}\, E'_{w^*}$*.*

Since for this case $C_p(E_w)$ is angelic, and then every compact subset of $C_p(E_w)$ is Fréchet-Urysohn, we note

Corollary 3

If E *is (WCG), then* B'_{w^*} *is Fréchet-Urysohn in the weak*-dual of* E*.*

Is E_w Lindelöf provided B'_{w^*} is Fréchet-Urysohn in the weak*-dual of E?

Example 4 (Corson-Pol)

Let D be the double arrow compact space and let $E := C(D)$. Then E_w is not Lindelöf but B'_{w^*} is Fréchet-Urysohn in the weak*-dual of E

The unit square $[0,1] \times [0,1]$ with the lexicographic order contains the double arrow space D. The space

$$D := ([0,1] \times \{0\}) \cup ([0,1] \times \{1\})$$

is compact non-metrizable and consists of the top and bottom edges of the unit square. D^2 is an example of a space that is normal that is not completely normal.

More about the space $C(K)$. The last space $C(D)$ enjoys the following property (so-called property (C) (**Corson**):

1. We say that $C(K)$ has property (C) if for every family $\mathcal{C}$ of closed convex subsets of $C(K)$ we have $\bigcap \mathcal{C} \neq \emptyset$ provided every countable subfamily of $\mathcal{C}$ has a nonempty intersection.
2. If $C(K)$ is weakly Lindelöf, then $C(K)$ has property (C).
3. Note that property (C) of E may be weakened to saying that E_w is realcompact.

Theorem 5 (Frankiewicz-Ryll-Plebanek)

If K is a compact space of countable tightness, then $C(K)_w$ is realcompact.

Theorem 6 (**Arkhangellski**)

If X is Lindelöf, then under PFA every compact subset of $C_p(X)$ has countable tightness.

Corollary 7

Assume E_w is Lindelöf. Under PFA the ball B'_{w^*} has countable tightness in the weak*-dual of E (since $B'_{w^*} \hookrightarrow C_p(B_w)$)

1. The question is if the converse leads property (C).
2. E has property (C) iff whenever $\mu \in \overline{A}$, where $A \subset B'_{w^*}$, there is countable $N \subset A$ with $\mu \in \overline{conv \ N}$ (**Pol**).
3. If K is compact zero-dimensional, E_w is Lindelöf for $E := C(K)$, then B'_{w^*} has countable tightness (**Frankiewicz, Plebanek, Ryll**), see works of Corson, Alster, Fremlin, Pol, Cascales, Plebanek, Ryll, etc.

Generalized metric concepts. Certain topological properties of function spaces have been intensively studied from many years. Particularly, various topological properties generalizing metrizability attracted specialists both from topology and analysis. One should mention, for example, Fréchet-Urysohn property, sequentiality, k-space property, $k_{\mathbb{R}}$-space property.

1. **metric $\Rightarrow$ first countable $\Rightarrow$ Fréchet-Urysohn $\Rightarrow$ sequential $\Rightarrow$ k-space $\Rightarrow$ $k_{\mathbb{R}}$-space .**
2. **sequential $\Rightarrow$ countable tightness.**

Theorem 8 (McCoy-Pytkeev)

For a completely regular Hausdorff space X the space $C_p(X)$ is Fréchet-Urysohn iff $C_p(X)$ is sequential iff $C_p(X)$ is a k-space. If X is a compact space, $C_p(X)$ is Fréchet-Urysohn iff X is scattered.

1. X is **sequential** if every sequentially closed subset of X is closed.
2. X is a k-**space** if for any Y any map $f : X \to Y$ is continuous, whenever $f|K$ for any compact K is continuous. X is a $k_{\mathbb{R}}$-**space** if the same holds for $Y = \mathbb{R}$.
3. X has **countable tightness** if whenever $x \in \overline{A}$ and $A \subseteq X$, then $x \in \overline{B}$ for some countable $B \subseteq A$.
4. X is **Fréchet-Urysohn** if whenever $x \in \overline{A}$ and $A \subseteq X$, there exists a sequence in A converging to x.

Problem 9

Characterize those Banach spaces E for which E_w (B_w, resp.) is a $k_{\mathbb{R}}$-space.

The space E_w is metrizable iff E is finite-dimensional BUT B_w is metrizable iff E' is separable (well-known!). Nevertheless, we have the following classical

Theorem 10 (Kaplansky)

If E is a metrizable lcs, E_w has countable tight.

Extension to class $\mathfrak{G}$ done by Cascales-Kakol-Saxon.

Theorem 11 (Schluchtermann-Wheller)

If E is a Banach space, then E_w is a k-space iff E is finite-dimensional.

The question when E_w is homeomorphic to a fixed model space from the infinite-dimensional topology is very restrictive and motivated specialists to detect above conditions only for some natural classes of subsets of E, e.g., ball B_w.

Theorem 12 (Schluchtermann-Wheller)

The following conditions are equivalent for a Banach space E: (a) B_w *is Fréchet–Urysohn;* (b) B_w *is sequential;* (c) B_w *is a* k*-space;* (d) E *contains no isomorphic copy of* ℓ_1.

Compare with the $C_p(X)$ case above!

Theorem 13 (Keller-Klee)

Let E be an infinite-dimensional separable reflexive Banach space. Then B_w is homeomorphic to $[0,1]^{\aleph_0}$.

Problem 14

Describe separable Banach spaces E with E_w homeomorphic.

If E and F are Banach spaces, E contains a copy of ℓ_1 and F does not, then E_w and F_w are not homeomorphic.

Ascoli spaces

1. A Tychonoff (Hausdorff) space X is called an **Ascoli space** if each compact subset K of $C_c(X)$ is evenly continuous.
2. Clearly: k-**space** $\Rightarrow$ $k_{\mathbb{R}}$-**space** $\Rightarrow$ **Ascoli space.**
3. For a topological space X, denote by $\psi : X \times C_c(X) \to \mathbb{R}$, $\psi(x, f) := f(x)$, the evaluation map. Recall that a subset K of $C_c(X)$ is evenly continuous if the restriction of ψ onto $X \times K$ is jointly continuous, i.e. for any $x \in X$, each $f \in K$ and every neighborhood $O_{f(x)} \subset Y$ of $f(x)$ there exist neighborhoods $U_f \subset K$ of f and $O_x \subset X$ of x such that $U_f(O_x) := \{g(y) : g \in U_f,\ y \in O_x\} \subset O_{f(x)}$.
4. A space X is Ascoli iff the canonical valuation map $X \hookrightarrow C_c(C_c(X))$ is an embedding (Gabriyelyan-Banakh).

Theorem 15 (Gabriyelyan-Kakol-Plebanek)

A Banach space E in the weak topology is Ascoli if and only if E is finite-dimensional.

Problem 16

Does there exist a Banach space E containing a copy of ℓ_1 such that B_w is Ascoli or a $k_{\mathbb{R}}$-space?

Theorem 17 (Gabriyelyan-Kakol-Plebanek)

The following are equivalent for a Banach space E.

(i) B_w *embeds into* $C_c(C_c(B_w))$;

(ii) B_w *is a* k*-space;*

(iii) B_w *is a* $k_{\mathbb{R}}$*-space;*

(iv) *a sequentially continuous real map on* B_w *is continuous;*

(v) E *does not contain a copy of* ℓ_1.

Let E be a Banach space containing a copy of ℓ_1 and again let B_w denote the unit ball in E equipped with the weak topology. We know already that B_w is not a $k_{\mathbb{R}}$-space iff there is a function $\Phi : B_w \to \mathbb{R}$ which is sequentially continuous but not continuous. How to construct such a function for E containing a copy of ℓ_1?

Recall that a (normalized) sequence (x_n) in a Banach space E is said to be equivalent to the standard basis of ℓ_1, or simply called an θ-ℓ_1-sequence, if for some $\theta > 0$

$$\left\| \sum_{i=1}^{n} c_i x_i \right\| \geq \theta \cdot \sum_{i=1}^{n} |c_i|,$$

for any natural number n and any scalars $c_i \in \mathbb{R}$.

Lemma 18 (Gabriyelyan-Kakol-Plebanek)

Let K be a compact space and let (g_n) be a normalized θ-ℓ_1-sequence in the Banach space $C(K)$. Then there exists a regular probability measure μ on K such that

$$\int_K |g_n - g_k| \, \mathrm{d}\mu \geq \theta/2 \textit{ whenever } n \neq k.$$

Example 19 (Gabriyelyan-Kakol-Plebanek)

Suppose that E is a Banach space containing an isomorphic copy of ℓ_1. Then there is a function $\Phi : B_w \to \mathbb{R}$ which is sequentially continuous but not continuous.

1. Let K denote the dual unit ball B_{E^*} equipped with the *weak** topology. Let Ix be the function on K given by $Ix(x^*) = x^*(x)$ for $x^* \in K$. Then $I : E \to C(K)$ is an isometric embedding.
2. Since E contains a copy of ℓ_1, there is a normalized sequence (x_n) in E which is a θ-ℓ_1-sequence for some $\theta > 0$. Then the functions $g_n = Ix_n$ form a θ-ℓ_1-sequence in $C(K)$. There is a probability measure μ on K such that $\int_K |g_n - g_k| \, \mathrm{d}\mu \geq \theta/2$ whenever $n \neq k$.
3. Define a function Φ on E by $\Phi(x) = \int_K |Ix| \, \mathrm{d}\mu$. If $y_j \to y$ weakly in E then $Iy_j \to Iy$ weakly in $C(K)$, i.e. $(Iy_j)_j$ is a uniformly bounded sequence converging pointwise to Iy. Consequently, $\Phi(y_j) \to \Phi(y)$ by the Lebesgue dominated convergence theorem. Thus Φ is sequentially continuous.

1. We now check that Φ is not weakly continuous at 0 on B_w. Consider a basic weak neighbourhood of $0 \in B_w$ of the form

$$V = \{x \in B_w : |x_j^*(x)| < \varepsilon \text{ for } j = 1, \ldots, r\}.$$

2. Then there is an infinite set $N \subset \mathbb{N}$ such that $\left(x_j^*(x_n)\right)_{n \in N}$ is a converging sequence for every $j \leq r$. Hence there are $n \neq k$ such that $|x_j^*(x_n - x_k)| < \varepsilon$ for every $j \leq r$, which means that $(x_n - x_k)/2 \in V$. On the other hand, $\Phi\left((x_n - x_k)/2\right) \geq \theta/4$ which demonstrates that Φ is not continuous at 0.

Theorem 20 (Pol)

For a metric separable space X the space $C_c(X)$ is a k-space iff X is locally compact.

Theorem 21 (Gabriyelyan-Kakol-Plebanek)

For first countable paracompact X, $C_c(X)$ is Ascoli iff $C_c(X)$ is a $k_{\mathbb{R}}$-space iff X is locally compact.

Corollary 22

For X first countable and paracompact and $C_p(X)$ angelic we have: $C_c(X)$ is a $k_{\mathbb{R}}$-space iff $C_c(X)$ is a Mazur space iff every sequentially continuous real map on $C_c(X)$ is continuous iff $C_c(X)$ is a Fréchet locally convex space.

The space $C_p([0,\omega_1))$ is angelic but $C_c([0,\omega_1))$ is not a $k_{\mathbb{R}}$-space; hence none of the above conditions holds.

It turns out that for metrizable spaces X we have a very concrete description of $k_{\mathbb{R}}$-spaces $C_c(X)$.

Theorem 23

For metrizable X the following are equivalent:

(i) $C_c(X)$ *is an Ascoli space.*

(i) $C_c(X)$ *is a Baire space.*

(ii) $C_c(X)$ *is a Baire-like space.*

(iii) $C_c(X)$ *is a $k_{\mathbb{R}}$-space.*

(iv) $C_c(X)$ *is the product Fréchet lcs.*

(v) X *is locally compact.*

*If X is additionally separable (**and complete**), the above are equivalent to: $C_c(X)$ is a separable Fréchet lcs (**and $C_c(X)$ has the strong Pytkeev property**).*

Characterize $k_{\mathbb{R}}$-spaces $C_p(X)$

Theorem 24

(i) *Let X be Čech-complete and Lindelöf. If $C_p(X)$ is a $k_{\mathbb{R}}$-space, then X is scattered and σ-compact.*

(ii) *If X is scattered and Lindelöf, then $C_p(X)$ is $k_{\mathbb{R}}$-space.*

Consequently, if X is a Čech-complete paracompact space, then $C_p(X)$ is a $k_{\mathbb{R}}$-space iff X is scattered and $C_p(X)$ is dominated by a metrizable vector topology. In particular, if X is a Čech-complete paracompact space, then $C_p(X)$ is a $k_{\mathbb{R}}$-space iff X is scattered.

Neither Čech-completeness nor paracompactness cannot be removed.

Problem 25

Characterize Ascoli spaces $C_p(X)$.

This yields also the following

Corollary 26

Let X be an analytic space. Then $C_p(X)$ is a $k_{\mathbb{R}}$-space iff $C_p(X)$ is metrizable, i.e. X is countable.

Note that "analyticity" cannot be replaced by "K-analyticity". Indeed, take a compact uncountable scattered space X, for example $X := [0, \omega_1]$.

Problem 27

Does there exist a Čech-complete Lindelöf space X such that $C_c(X)$ is a non metrizable $k_{\mathbb{R}}$-space (equivalently, not being the product of metrizable spaces)?

Networks for the weak topology of Banach spaces.

A (WCG) Banach space E is separable iff every compact set in E_w is metrizable iff E'_{w^*} is separable. What about nonseparable Banach spaces E for which all compact sets in E_w are metrizable?

1. A space X is a continuous image under a compact-covering map from a metrizable space Y iff every compact set in X is metrizable (**Michael**).
2. A regular space X has a countable k-network iff X is a continuous image under a compact-covering map from a metrizable separable space Y (**Michael**)
3. A continuous map $f : Y \to X$ is compact-covering if for every compact $K \subset X$ there exists compact $L \subset Y$ such that $f(L) = K$.

1. For many classes of (separable) Banach spaces E, the space E_w is a generalized metric space of some type. Such types of spaces are defined by different types of networks. The concept of network, coming from the pure set-topology, which turned out to be of great importance to study successfully renorming theory in Banach spaces, see the survey paper **Cascales-Orihuela**.
2. A regular space X is an $\aleph_0$-space (**Michael**) if it has a **countable** k-**network**, i.e. there is a countable family $\mathcal{D}$ of subsets of X if whenever $K \subset U$ with K compact and U open in X, there exists $D \in \mathcal{D}$ with $K \subset D \subset U$. Any separable metric space is an $\aleph_0$-space (by **Urysohn's** theorem).

1. **O'Meara** generalized the concept of $\aleph_0$-spaces as follows: A topological space X is called an $\aleph$-space if it is regular and has a σ-**locally finite** k-**network**. Any metrizable space is an $\aleph$-space (by **Nagata-Smirnov's** theorem) and all compact sets in $\aleph$-spaces are metrizable, see papers of **Gruenhage**.
2. A regular space X is a σ-space if it has a σ-locally finite network.
3. A regular space X is an $\aleph_0$-space (resp. a σ-space) iff X is Lindelöf and is an $\aleph$-space (resp. has a countable network) [**S-Ka-Ku-M**].

Problem 28

Characterize those Banach spaces E for which E_w admits a "good type" of network.

Theorem 29 (Gabriyelyan-Kakol-Kubis-Marciszewski)

The Banach space $\ell_1(\Gamma)$ is an $\aleph$-space in the weak topology iff the cardinality of Γ does not exceed the continuum.

1. Hence $\ell_1(\mathbb{R})$ endowed with the weak topology is an $\aleph$-space but it is not an $\aleph_0$-space (as nonseparable) and $\ell_1(\mathbb{R})$ in the weak topology is not normal.
2. If K is compact, the Banach space $C(K)$ is a weakly $\aleph_0$-space if and only if K is countable (**Corson**). The following general theorem extends this result.

Theorem 30 (Gabriyelyan-Kakol-Kubis-Marciszewski)

A Fréchet lcs $C_c(X)$ is a weakly $\aleph$-space if and only if $C_c(X)$ is a weakly $\aleph_0$-space if and only if X is countable.

Last Theorem combined with some recent results of **Pol-Marciszewski** provides concrete Banach spaces $C(K)$ which under the weak topology are σ-spaces but not $\aleph$-spaces.

Corollary 31 (Gabriyelyan-Kakol-Kubis-Marciszewski)

Let K be an uncountable separable compact space. If K is a linearly ordered space, or a dyadic space, then $C(K)$ endowed with the weak topology is a σ-space but not an $\aleph$-space. If additionally K is metrizable, then $C(K)$ endowed with the weak topology is a cosmic space but is not an $\aleph$-space.

Theorem 32 (Gabriyelyan-Kakol-Kubis-Marciszewski)

Let E be a Banach space not containing a copy of ℓ_1. Then E is a weakly $\aleph$-space if and only if E is a weakly $\aleph_0$-space if and only its strong dual E' is separable.

Networks and renormings. Let E be a Banach space endowed with a norm $\|.\|$ and let S be the unit sphere. Then the norm is said to be

1. *locally uniformly rotund* (LUR), if, whenever $x, x_k \in E$ are such that $\|x\| = \|x_k\| = 1$ and $\|x_k + x\| \to 2$, then $\|x - x_k\| \to 0$
2. *Kadec* , if $\sigma(E, E')$ and the norm topology coincide on S.
3. The space $\ell_1([0,1])$ has in the weak topology a σ-locally finite k-network and yet has an equivalent (LUR)-norm.

Theorem 33 (Troyanski)

E_w has an equivalent norm for (WCG) Banach spaces E.

Problem 34

Does every Banach space E for which E_w has a σ-locally finite k-network have an equivalent (LUR)-norm?

1. Questions concerning renormings in Banach spaces have been of particular importance to provide smooth functions and tools for optimization theory. An excellent monograph of renorming theory up to 1993 by **R. Deville, G. Godefroy**, and **V. Zizler**.
2. If E admits a (LUR) norm as well as another norm whose dual norm is (LUR), then E admits a third norm which is simultaneously C^1-smooth and (LUR)(**Asplund**) .
3. If E is a (WCG) Banach space admitting a C^k-smooth norm where $k \in \mathbb{N} \cup \{\infty\}$, then E admits an equivalent norm which is simultaneously, C^1-smooth, (LUR), and the limit of a sequence of C^k-smooth norms (**Hajek**).
4. A dual Banach space E' has an equivalent (W^*LUR) norm iff the weak*-topology has a σ-isolated network (**Raja**).

1. A Banach space with a (LUR)-norm has a Kadec norm.
2. It seems that nothing is known about the relation between the property of being a weakly $\aleph$-space and the existence of a (LUR) norm on a Banach space. Another motivation for above problems might be related with the following result.

Recall that E has a $\sigma(E,E')$-LUR norm $\|.\|$ if $x_k \to x$ in $\sigma(E,E')$, whenever $(2\|x\|^2 + 2\|x_k\|^2 - \|x + x_k\|^2) \to 0$.

Theorem 35 (Molto-Orihuela-Troyanski-Valdivia)

If E is a Banach space, then E admits an equivalent $\sigma(E,E')$-lower semicontinuous and $\sigma(E,E')$-LUR norm iff $\sigma(E,E')$ admits a network $\mathcal{N} = \bigcup_n N_n$, where N_n is $\sigma(E,E')$-slicely isolated. A Banach space E has a $\sigma(E,E')$-LUR norm iff it has an equivalent (LUR) norm.

A problem about the Dugundji Extension Property for B_w.

1. A space X is said to have the Dugundji Extension Property (DEP) if, for every closed $A \subset X$, there is a continuous linear extender $e : C(A) \to C(X)$, $e(f)|A = f$ for any $f \in C(A)$, where both spaces $C(A)$ and $C(X)$ are with the compact-open topology.
2. Metrizable spaces have the (DEP).Stratifiable spaces (in sense of Borges) have the (DEP).
3. If H is a non-separable Hilbert space, then its B_w does not have the (DEP)(**Avilles-Marciszewski**).

Problem 36

Characterize those Banach spaces E for which B_w has the (DEP).