Topological groups and C(X) spaces with ordered bases

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Outline

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1. Σ-bases in topological groups
   - $\mathfrak{G}$-bases and quasi-$\mathfrak{G}$-bases
   - Σ-bases and $C_c(X)$ with Σ-base

$\mathfrak{G}$-bases

Definition

A topological group G is said to have a $\mathfrak{G}$-base if there is a base $\{U_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ of neighborhoods of the identity e in G such that $U_\beta \subseteq U_\alpha$ whenever $\alpha \leq \beta$.

- Metrizable topological group $\Longrightarrow$ $\mathfrak{G}$-base.
- Fréchet-Urysohn topological group with a $\mathfrak{G}$-base $\Longrightarrow$ metrizable (Grabriyelyan ..., Fundamenta Math. 2015).

Definition

A compact resolution on a topological space X is a compact covering $\mathcal{K} = \{K_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ of X such that $K_\alpha \subseteq K_\beta$ whenever $\alpha \leq \beta$. If for each compact subset K of X there exists K_α such that $K \subset K_\alpha$, then $\mathcal{K}$ is a compact resolution swallowing compact subsets.

$\mathfrak{G}$-bases in $C_c(X)$

Theorem

A space $C_c(X)$ has a $\mathfrak{G}$-base $\{U_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ of (absolutely convex) neighborhoods of the origin if and only if X has a compact resolution $\mathcal{K} = \{K_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ swallowing compact subsets

$\mathfrak{G}$-bases in $C_c(X)$

Proof.

We may suppose that there exists a compact K

$$U_\alpha \subset W(K,[-1,1]) := \{f \in C_c(X) : f(K) \subset [-1,1]\}, \quad \text{hence}$$

$$K \subset K_\alpha := \cap_{f\in U_\alpha} f^{-1}([-1,1]) \text{ and } U_\alpha \subset W(K_\alpha,[-1,1]),\ \alpha \in \mathbb{N}^{\mathbb{N}}$$

There exists a compact K_{U_α} and $\varepsilon_\alpha > 0$ such that $W(K_{U_\alpha},(-\varepsilon,\varepsilon)) \subset U_\alpha$. Then

$$W(K_{U_\alpha},(-\varepsilon,\varepsilon)) \subset W(K_\alpha,[-1,1]) \implies K_\alpha \subset K_{U_\alpha}.$$

$\{K_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a compact resolution swallowing compact sets
The converse follows from

$$W(K_{\alpha=(a_1,..)},[-a_1^{-1},a_1^{-1}]) \subset W(K,[-\varepsilon,\varepsilon]) \quad \text{if} \quad K \subset K_\alpha,\ a_1^{-1} < \varepsilon.$$

$\mathfrak{G}$-bases in non-metrizable $C_c(X)$

Corollary

If X is a Polish space the $C_c(X)$ has a $\mathfrak{G}$-base. Whence $C_c(\mathbb{R}^{\mathbb{N}})$ is a non-metrizable locally convex space with a $\mathfrak{G}$-base.

Proof.

Let $\{x_n : n \in \mathbb{N}^{\mathbb{N}}\}$ be a dense subset of X, d a complete metric compatible and $B(x_{a_m}, n^{-1})$ the closed ball of center x_{a_m} and radius n^{-1}. If $\alpha := (a_n)_n \in \mathbb{N}^{\mathbb{N}}$ and

$$K_\alpha := \cap_{n \in \mathbb{N}^{\mathbb{N}}} [\cup_{1 \leq m \leq n} B(x_{a_m}, n^{-1})]$$

we get that $\{K_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a compact resolution of X swallowing compact sets.
Finally, $\mathbb{R}^{\mathbb{N}}$ is Polish but not hemicompact. □

Strong Pytkeev property and quasi-$\mathfrak{G}$-bases

Definition (Tsaban and Zdomskyy, 2009)

A topological group G has the strong Pytkeev property if there exists a sequence $\mathcal{D}$ of subsets of G satisfying the property: for each neighborhood U of the unit e and each $A \subseteq G$ with $e \in \overline{A} \setminus A$, there is $D \in \mathcal{D}$ such that $D \subseteq U$ and $D \cap A$ is infinite.

Proposition (Gabriyelyan, Kąkol and Leiderman, 2014)

Any topological group G with the strong Pytkeev property admits a quasi-$\mathfrak{G}$-base $\{U_\alpha : \alpha \in \Sigma\}$ of the identity, i.e., an ordered base of neighborhoods $\{U_\alpha : \alpha \in \Sigma\}$ of e over some $\Sigma \subseteq \mathbb{N}^{\mathbb{N}}$.

Strong Pytkeev property and quasi-$\mathfrak{G}$-bases

Proposition (Banakh, 2015)

For every separable metrizable space X the space $C_c(X)$ has the strong Pytkeev property; therefore such $C_c(X)$ admits a quasi-$\mathfrak{G}$-base.

Remark

Let X be a separable metric space which is not a Polish space. Then $C_c(X)$ has a quasi-$\mathfrak{G}$-base but $C_c(X)$ does not admit a $\mathfrak{G}$-base.

$C_c(X)$ with Σ-base

The following is a is more practical concept than quasi-$\mathfrak{G}$-base.

Definition

If $\Sigma \subseteq \mathbb{N}^{\mathbb{N}}$ is an unbounded (i.e., $\sup\{\alpha(k) : \alpha \in \Sigma\} = \infty$ for some $k \in \mathbb{N}$) and directed subset of $\mathbb{N}^{\mathbb{N}}$, a base $\{U_\alpha : \alpha \in \Sigma\}$ of neighborhoods of the neutral element of a topological group G is a Σ-base if $U_\beta \subseteq U_\alpha$ whenever $\alpha \leq \beta$ with $\alpha, \beta \in \Sigma$.

Theorem (For a completely regular space X are equivalent:)

1. *The locally convex space $C_c(X)$ has a Σ-base of absolutely convex neighborhoods of the origin.*
2. *There is a compact covering $\{K_\alpha : \alpha \in \Sigma\}$ of X that swallows the compact sets of X, with Σ unbounded, directed and such that $K_\alpha \subseteq K_\beta$ whenever $\alpha \leq \beta$ in Σ.*

$C_c(X)$ with Σ-base

Proof.

If U_α is a neighborhood of the origin in $C_c(X)$ and K is a compact subset of X such that

$$U_\alpha \subset W(K,[-1,1]) := \{f \in C_c(X) : f(K) \subset [-1,1]\},$$

then

$$K \subset K_\alpha := \cap_{f\in U_\alpha} f^{-1}([-1,1]) \quad \text{and} \quad U_\alpha \subset W(K_\alpha,[-1,1]).$$

Let K_{U_α} be a compact set such that $W(K_{U_\alpha},(-\varepsilon,\varepsilon)) \subset U_\alpha$. Then

$$W(K_{U_\alpha},(-\varepsilon,\varepsilon)) \subset W(K_\alpha,[-1,1]) \implies K_\alpha \subset K_{U_\alpha}.$$

□

$C_c(X)$ with Σ-base

continued proof.

If $C_c(X)$ has a Σ-base there exists a compact subset K of X and a Σ-base $\{U_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ such that $U_\alpha \subset W(K, [-1, 1])$, for each $\alpha \in \mathbb{N}^\mathbb{N}$. Whence $\{K_\alpha : \alpha \in \Sigma\}$ is a compact covering of X, with Σ unbounded and directed, such that $K_\alpha \subseteq K_\beta$ whenever $\alpha \leq \beta$ in Σ, that swallows the compact sets.
To proof the converse we must to take into account that given a compact subset K of X and a positive real number $\varepsilon > 0$ there exists $\alpha \in \Sigma$ such that $K \subset K_\alpha$ and $a_n^{-1} < \varepsilon$, whence

$$W(K_{\alpha=(a_1,..)}, [-a_n^{-1}, a_n^{-1}]) \subset W(K, [-\varepsilon, \varepsilon]).$$

□

Σ-base $\not\Longrightarrow$ $\mathfrak{G}$-base

Theorem

If (X,d) is a separable and not Polish, then $C_c(X)$ admits Σ-base and it does not admit any $\mathfrak{G}$-base.

Proof (only the non trivial part).

Let $D := \{x_m : m \in \mathbb{N}\}$ dense subset in X, $\{y_n : n \in \mathbb{N}\}$ dense in K (compact), $x_{n(p)} \in D$ with

$$\lim_p x_{np} = y_n \quad \text{and} \quad d(x_{np}, y_n) < n^{-1}, \text{ for each } p \in \mathbb{N},$$

then $K \subset \overline{\{x_{np} : (n,p) \in \mathbb{N}^2\}}$ (compact). The Σ-base follows from the set Σ of $\alpha := (a_n)_n \in \cup_{m\in\mathbb{N}\setminus\{1\}}\{1,m\}^{\mathbb{N}}$ with compact

$$K_\alpha := \overline{\{x_n : n \in \mathbb{N},\ a_n \neq 1\}}$$

Outline

2. Boundedly complete sets and long Σ-bases
   - Boundedly complete subsets of $\mathbb{N}^{\mathbb{N}}$
   - Long Σ-bases

Boundedly complete sets in $\mathbb{N}^{\mathbb{N}}$

In this section we are going to consider a special class of Σ-bases, which we denominate long Σ-bases, and study some properties of them quite close to those of $\mathfrak{G}$-bases.

Definition

A subset Σ of $\mathbb{N}^{\mathbb{N}}$ will be called boundedly complete if each bounded set Δ of Σ has a bound at Σ.

- Σ boundedly complete $\Longrightarrow$ Σ is directed.
- If $\{U_\alpha : \alpha \in \Sigma\}$ is an infinite base of neighborhoods of a (Hausdorff) locally convex space and Σ is a boundedly complete subset of $\mathbb{N}^{\mathbb{N}}$ then Σ must be unbounded. (Otherwise $\sup\{\alpha(k) : \alpha \in \Sigma\} < \infty$ for every $k \in \mathbb{N} \Longrightarrow$ there exists $\gamma \in \Sigma$ with $\alpha \leq \gamma$ for every $\alpha \in \Sigma$. Hence

$$U_\gamma \subseteq \cap_{\alpha\in\Sigma} U_\alpha,\text{ a contradiction)}$$

Compact coverings and strong domination

Example

Every cofinal subset Σ of $\mathbb{N}^{\mathbb{N}}$ with respect to the partial order '$\leq$' is boundedly complete.

Proof.

If $\beta(k) := \sup\{\alpha(k) : \alpha \in \Delta\} < \infty$ for every $k \in N$, then $\beta := (\beta(k))_k \in N^{\mathbb{N}}$, hence there is $\gamma \in \Sigma$ such that $\beta \leq \gamma$. □

Proposition

If X is a topological space with a compact covering $\{A_\alpha : \alpha \in \Sigma\}$ that swallows the compact sets indexed by a boundedly complete subset Σ of $\mathbb{N}^{\mathbb{N}}$ and such that $A_\alpha \subseteq A_\beta$ whenever $\alpha \leq \beta$ in Σ, then X is strongly dominated by a second countable space.

Compact coverings and strong domination

Proof.

Let $T : \Sigma \to \mathcal{K}(X)$ defined by $T(\alpha) = A_\alpha$ and let K be a compact set in Σ.

$$\sup\{\alpha(k) : \alpha \in K\} < \infty, \forall\, k \in \mathbb{N} \Longrightarrow \exists\gamma \in \Sigma\,,\, \alpha \leq \gamma, \forall\alpha \in K.$$

$$T(K) = \cup_{\alpha\in K} T(\alpha) \subseteq A_\gamma \Longrightarrow B_K := \overline{T(K)} \text{ is compact}$$

$\mathcal{B} := \{B_K : K \in \mathcal{K}(\Sigma)\}$ is an increasing compact covering of X that swallows the compact sets, because

$$\text{if } P \text{ is compact } \exists\delta \in \Sigma \text{ with } P \subseteq T(\delta) = B_{\{\delta\}}.$$

Hence X is strongly Σ-dominated (Σ separable metric). □

Long Σ-bases

Definition

A Σ-base of neighborhoods of the unit element of a topological group G indexed by a boundedly complete subspace Σ of $\mathbb{N}^{\mathbb{N}}$ will be referred to as a long Σ-base.

Of course, every $\mathfrak{G}$-base of neighborhoods of the origin of a locally convex space E is a long Σ-base, with $\Sigma = \mathbb{N}^{\mathbb{N}}$. The proof of the next theorem uses the following

Proposition (Cascales, Orihuela, Tkachuk, 2011)

A compact topological space K is metrizable if and only if the space $(K \times K) \setminus \Delta$ is strongly dominated by a second countable space, where here $\Delta := \{(x,x) : x \in K\}$.

Long Σ-bases and metrizability

Theorem

If a topological group G has a long Σ-base $\{U_\alpha : \alpha \in \Sigma\}$ then every compact subset K in G is metrizable. Consequently, G is strictly angelic.

Proof.

It is enough to show that $W := (K \times K) \setminus \Delta$ has a compact covering $\mathcal{W} := \{W_\alpha : \alpha \in \Sigma\}$ that swallows the compact sets indexed by a boundedly complete subset $\Sigma \subseteq \mathbb{N}^{\mathbb{N}}$ and such that $W_\alpha \subseteq W_\beta$ whenever $\alpha \leq \beta$ in Σ. We may assume that all sets U_α are symmetric and open. Then □

Long Σ-bases and metrizability

continued proof.

$$W_\alpha := \{(x,y) \in W : xy^{-1} \notin U_\alpha\}$$

- is closed in $K \times K$, hence W_α compact. If $Q \subseteq W$ is a compact set. Then

  $$e \notin T(Q) := \{xy^{-1} : (x,y) \in Q\} \text{ (compact)},$$

  implies there exists U_α such that

  $$U_\alpha \cap T(Q) = \emptyset \Longrightarrow Q \subseteq W_\alpha.$$

- $\mathcal{W} := \{W_\alpha : \alpha \in \Sigma\}$ verifies the conditions.

□

Angelicity $C_c(X)$

Corollary

If there exists a family $\{A_\alpha : \alpha \in \Sigma\}$ made up of compact sets, indexed by a boundedly complete set Σ such that $A_\alpha \subseteq A_\beta$ whenever $\alpha \leq \beta$ and satisfying that $\overline{\cup\{A_\alpha : \alpha \in \Sigma\}} = X$, then $C_c(X)$ is strictly angelic.

Proof.

X is web-compact, so $C_p(X)$ is angelic (Orihuela 1987), whence $C_c(X)$ is angelic (by angelic lemma). To prove "strict" let $Y = \cup\{A_\alpha : \alpha \in \Sigma\}$ and τ_p and τ_c pointwise and the compact-open topology on $C(Y)$. □

Angelicity $C_c(X)$

continued proof.

(Σ boundedly complete $\Longrightarrow$ unbounded in $\mathbb{N}^{\mathbb{N}}$ $\Longrightarrow$) there exists $k \in \mathbb{N}$ such that $\sup\{\alpha(k) : \alpha \in \Sigma\} = \infty$. Then

$$\{U_\alpha : \alpha \in \Sigma\}, \text{ with } U_\alpha := \{f \in C(Y) : \sup_{y \in A_\alpha} |f(y)| \leq \alpha(k)^{-1}\}$$

is a long Σ-base of a lc topology τ on $C(Y)$ such that $\tau_p \leq \tau \leq \tau_c$. By preceding Theorem every τ-compact set in $C(Y)$ is metrizable. Whence each compact subset K of $C_c(X)$ is metrizable since the restriction map $S : C_c(X) \to (C(Y), \tau)$ is continuous and S restricts itself to an homeomorphism on each compact subset K of $C_c(X)$. □

$C_c(X)$ with a long Σ-base

Theorem

If $C_c(X)$ has a long Σ-base of neighborhoods of the origin, then X is a C–Suslin space. Consequently $C_c(X)$ is angelic.

Proof.

X has a compact covering $\{K_\alpha : \alpha \in \Sigma\}$ swallowing compacts such that $K_\alpha \subseteq K_\beta$ whenever $\alpha \leq \beta$.
Let $T : \Sigma \to \mathcal{K}(X)$ defined by $T(\alpha) = A_\alpha$.
If $\alpha_n \in \Sigma$ and $\lim_n \alpha_n = \alpha \in \mathbb{N}^{\mathbb{N}}$, then there is $\gamma \in \Sigma$ with $\alpha_n \leq \gamma$ for every $n \in \mathbb{N}$.
Consequently, $\{T(\alpha_n) : n \in \mathbb{N}\} \subset A_\gamma$. Hence $x_n \in T(\alpha_n)$, $\forall n \in \mathbb{N}$, $\Longrightarrow \{x_n\}_{n=1}^{\infty}$ has a cluster point x in X (contained in A_γ).
Therefore X is web-compact. □

A limit property in Fréchet-Urysohn topological groups

Let $\{U_\alpha : \alpha \in \Sigma\}$ be a long Σ-base in a topological group G. For every $\alpha = (a_i)_{i\in\mathbb{N}} \in \Sigma$ and each $k \in \mathbb{N}$, set

$$\alpha(k) := (a_1, a_2, \ldots, a_k)$$

$$D_k(\alpha) := \cap\{U_\beta : \beta \in \Sigma, \;\; \beta(k) = \alpha(k)\}.$$

Clearly, $\{D_k(\alpha)\}_{k\in\mathbb{N}}$ is an increasing and $e \in D_k(\alpha)$.

Proposition (Chasco, Martín-Peinador and Tarieladze, 2007)

Let $\{x_{n,k} : (n,k) \in \mathbb{N}\times\mathbb{N}\}$ a subset of a Fréchet-Urysohn topological group G such that $\lim_n x_{n,k} = x \in G$, $k = 1, 2, \ldots$. There exists two increasing sequences of natural numbers $(n_i)_{i\in\mathbb{N}}$ and $(k_i)_{i\in\mathbb{N}}$, such that $\lim_i x_{n_i,k_i} = x$.

Metrizability in Fréchet-Urysohn topological groups

Theorem

Each Fréchet-Urysohn topological group G with a long Σ-base $\{U_\alpha : \alpha \in \Sigma\}$ is metrizable.

Proof.

Assume $\exists \alpha \in \Sigma$ such that $D_k(\alpha)$ is not a neighborhood of the unit e for every $k \in \mathbb{N}$.
$e \in \overline{G \setminus D_k(\alpha)} \Longrightarrow \exists \{x_{n,k}\}_{n\in\mathbb{N}}$ in $G \setminus D_k(\alpha)$ converging to e.
Hence exists $(n_i)_{i\in\mathbb{N}} \uparrow$ and $(k_i)_{i\in\mathbb{N}} \uparrow$ such that $\lim_i x_{n_i,k_i} = e$.
$x_{n_i,k_i} \notin D_{k_i}(\alpha) \Longrightarrow \exists \beta_{k_i} \in \Sigma,\ \beta_{k_i}(k_i) = \alpha(k_i),\ x_{n_i,k_i} \notin U_{\beta_{k_i}}$.
$x_{n_i,k_i} \notin U_\gamma$ for every $i \in \mathbb{N}$, if $\beta_{k_i} \leqslant \gamma$, $i \in \mathcal{N}$. Contradiction.
For every $\alpha \in \Sigma$ choose the minimal $k_\alpha \in \mathbb{N}$ such that $D_{k_\alpha}(\alpha)$ is a neighborhood of e.
$\{\operatorname{int}(D_{k_\alpha}(\alpha))\}_{\alpha\in\Sigma}$ is base of neigh. of e, so G is metrizable. □

Long Σ-bases in products

Corollary

Let $\{G_t)_{t\in T}$ be a family of metrizable topological groups. Then the product $G := \prod_{t\in T} G_t$ has a long Σ-base if and only if T is countable, i.e., when G is metrizable.

Proof.

Let e_t be the unit vector in G_t for $t \in T$.
The Σ-product $G_0 := \{x = (x_t) \in G : |t \in T : x_t \neq e_t| \leq \aleph_0\}$ is a dense Fréchet-Urysohn subgroup of G (Noble, 1970).
If G has a long Σ-base, then G_0 enjoys also this property.
Whence G_0 is metrizable, so G is metrizable, too. The converse is clear. $\square$

Long Σ-bases in $C_p(X)$

Corollary

The space $C_p(X)$ has a long Σ-base if and only if X is countable.

Proof.

Apply preceding Corollary to $\mathbb{R}^X = \overline{C_p(X)}$ $\square$

Outline

3. Existence of proper long Σ-bases on $C_c([0, \omega_1))$

The dominating cardinal

In $(\mathbb{N}^{\mathbb{N}}, \leq^*)$

- $\alpha \leq^* \beta$ stands for the *eventual dominance preorder* defined so that $\alpha(n) \leq \beta(n)$ for almost all $n \in \mathbb{N}$, i.e., for all but finitely many values of n.
- $\alpha <^* \beta$ means that there exists $m \in \mathbb{N}$ such that $\alpha(n) < \beta(n)$ for every $n \geq m$.

ω_1 is the first ordinal of uncountable cardinal, whose cardinality we denote by $\aleph_1$.
ZFC model means Zermelo–Fraenkel model + axiom of choice.

Definition

The *dominating cardinal* $\mathfrak{d}$ is the least cardinality for cofinal subsets of the preordered space $(\mathbb{N}^{\mathbb{N}}, \leq^*)$.

One has $\aleph_1 \leq \mathfrak{d} \leq \mathfrak{c}$.

The main lemma

Lemma

If $\aleph_1 = \mathfrak{d}$ *there exists a cofinal* ω_1*-sequence* $\Gamma := \{\beta_\kappa : \kappa < \omega_1\}$ *in* $(\mathbb{N}^\mathbb{N}, \leq^*)$ *such that*

1. $\kappa_1 < \kappa_2$ *implies that* $\beta_{\kappa_1} <^* \beta_{\kappa_2}$*,*
2. *for each* $\alpha \in \mathbb{N}^\mathbb{N}$ *the subset*
$$\Delta_\alpha := \{\kappa < \omega_1 : \beta_\kappa \leq^* \alpha\}$$
of $[0,\omega_1)$ *is countable,*
3. *if* $\alpha \leq^* \gamma$ *then* $\Delta_\alpha \subseteq \Delta_\gamma$*, and*
4. *every countable subset of* $[0,\omega_1)$ *is contained in some* Δ_γ*; in particular,* $\bigcup_{\alpha\in\mathbb{N}^\mathbb{N}} \Delta_\alpha = [0,\omega_1)$*.*

Example

Example

In any ZFC model for which $\aleph_1 = \mathfrak{d} < \mathfrak{c}$ there exists a completely regular space X and a compact covering $\{A_\alpha : \alpha \in \Sigma\}$ of X, with $A_\alpha \subseteq A_\beta$ whenever $\alpha \leq \beta$ and indexed by an unbounded, directed and boundedly complete proper subset Σ of $\mathbb{N}^{\mathbb{N}}$ that swallows the compact sets of X.

Corollary

In any ZFC model for which $\aleph_1 = \mathfrak{d} < \mathfrak{c}$ there exists a long Σ-base of absolutely convex neighborhoods of the origin of the space $C_c([0,\omega_1))$ which is not a $\mathfrak{G}$-base.

Open question

Problem

Let X be a separable metric space admitting a compact ordered covering of X indexed by an unbounded and boundedly complete proper subset of $\mathbb{N}^{\mathbb{N}}$ that swallows the compact sets of X. Is then X a Polish space?

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